

# STOKES FLOW WITH KINEMATIC AND DYNAMIC BOUNDARY CONDITIONS

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ABSTRACT. We study the solvability of the Stokes system in a bounded Lipschitz domain in  $\mathbb{R}^n$  when the flow is subjected to a mixed boundary condition: a Dirichlet condition for the velocity and a Neumann condition for the stress tensor. Some minor modifications to the standard theory are therefore required. The most noteworthy result is that both pressure and velocity are unique.

## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  ( $n \geq 2$ ), whose boundary  $\partial\Omega$ , with outward unit normal  $\hat{n}$ , is divided into two disjoint parts:  $\Gamma_D$  and  $\Gamma_N$ . The present work concerns the solvability of the boundary value problem (b.v.p.)

$$(1) \quad \begin{cases} \operatorname{div} \sigma = f & \text{in } \Omega & (1a) \\ \operatorname{div} u = 0 & \text{in } \Omega & (1b) \\ \sigma \hat{n} = g & \text{on } \Gamma_N & (1c) \\ u = h & \text{on } \Gamma_D & (1d). \end{cases}$$

The symbol  $\sigma = (\sigma_{ij})_{ij}$  denotes the Cauchy stress tensor which is defined by the constitutive law

$$(2) \quad \sigma_{ij} = -p \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (1 \leq i, j \leq n),$$

where the dynamic viscosity  $\mu > 0$  is a given constant. Under this assumption, (1a–b) becomes the Stokes system which describes the steady flow of an incompressible Newtonian fluid when inertial forces can be neglected. Moreover, we assume that  $f$  (body forces),  $g$  (surface forces) and  $h$  (surface velocity) are given vector functions and we seek the velocity  $u$  (vector function) and the pressure  $p$  (scalar function) of the fluid.

The b.v.p. (1) when  $\Gamma_D = \partial\Omega$  and  $\Gamma_N = \emptyset$  is classical and has been studied by many authors. The main novelty of the present work is the mixed boundary condition (1c–d). The Dirichlet condition (1d) is a *kinematic* condition that prescribes the velocities of fluid particles at

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each point of the surface  $\Gamma_D$ . Such a condition is natural if  $\Gamma_D$  is the contact surface between a fluid and a body moving with a prescribed velocity, provided that the velocity vector is continuous across the interface. This so called no-slip condition is valid at solid boundaries but also at the interface between two similar fluids [18, pp. 12–14]. We do not assume that  $\Gamma_D$  is impermeable, so  $h$  may have a normal component — fluid particles may enter or leave the domain through any part of the boundary as long as (1b) is fulfilled. The Neumann condition (1c), on the other hand, is a *dynamic* condition that prescribes the stress vector (traction) at each point of  $\Gamma_N$ . In physical terms, we say that the fluid is subjected to an externally applied surface force as a result of the mechanical contact with a surrounding body. The dynamic condition is applicable whenever the stress vector is continuous across a surface. In particular, this is true when  $\Gamma_N$  is the interface between two fluids (not necessarily of the same constitutive type), see [10, Art. 327] or [19, p. 240]. For a thorough discussion of fluid-fluid interfaces we refer to Chapter 5 of [18].

It is instructive to consider (1) under the assumption that  $\Omega$  is surrounded by a fluid of similar type, i.e. one satisfying (1a-b) in the exterior domain. Suppose that  $g$  (stress vector) and  $h$  (velocity vector) are the observed boundary values of the surrounding fluid. Would it then be possible to extend the velocity and the pressure of this fluid in a unique way into  $\Omega$  so that both quantities are continuous across the boundary? Our main results asserts that the answer is affirmative.

The kinematic condition is treated in all classical texts devoted to the mathematical study of fluids [6, 9, 11, 22]. But there are very few works related to dynamic condition in the mathematical literature. This cannot be said about the related field of elasticity theory where equal attention is given to both conditions (see e.g. [5, 17]). A strong argument for considering the mixed condition for fluids comes from the solvability of (1) alone. Under fairly general assumptions on  $f, g$  and  $h$  it can be shown that the b.v.p. (1) has a weak solution:

- (i) If  $\int_{\Gamma_N} dS = 0$  and  $\int_{\partial\Omega} h \cdot \hat{n} dS = 0$ , there exists a solution  $(u, p)$  of (1). The velocity is unique but the pressure is only unique up to a constant.
- (ii) If  $\int_{\Gamma_D} dS = 0$  and  $f$  is “compatible” with  $g$ , there exists a solution  $(u, p)$  of (1). The pressure is unique but the velocity is only unique up to the motion of a rigid body.
- (iii) If  $\int_{\Gamma_D} dS > 0$  and  $\int_{\Gamma_N} dS > 0$ , there exists a solution  $(u, p)$  of (1) for “arbitrary” data  $f, g$  and  $h$ . Both the velocity and the pressure are unique.

This result has practical implications. In many applications involving viscous flows, e.g. in lubrication theory, the main problem is to calculate the net force exerted by the fluid on a solid boundary. For such

problems the kinematic condition is obviously deficient, inasmuch as the dynamic state of the fluid cannot be completely determined. A remedy is to impose a dynamic condition on some part of the boundary. On a more general note, if one believes that the Stokes system (1a–b) is a good model for the creeping motion of a viscous fluid in the real world, where both velocity and stress are observable quantities, the mixed condition seems like the most realistic alternative.

A precise formulation of the solvability of (1) when  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  and  $\partial\Omega$  is of class  $C^{0,1}$ , is the principal result of the present work. For the Dirichlet problem, one deduces the existence of a pressure function from De Rham’s theorem. For a simple proof of this result we refer to Tartar [20, 21]. We would like to emphasise, that the mixed boundary condition requires a *stronger* formulation of this result. Nevertheless, the dynamic condition has been studied by some authors. The solvability of (1) when  $n = 3$  and  $\Gamma_N = \partial\Omega$  is of class  $C^{1,1}$  is studied by Boyer and Fabrie in Chapter IV of [2] (without references to previous work). The authors observed that the pressure is unique, but did not consider the mixed boundary condition. Maz’ya and Rossmann [13, 14] have studied the stationary Stokes system in a three-dimensional domain of polyhedral type with one of four different boundary conditions, including (1c–d), imposed on each face. Their investigations are mainly concerned with regularity assertions of weak solutions, but existence and uniqueness of the mixed boundary value problem for the Stokes system is proved in Theorem 5.1 of [13]. The present paper extends the aforementioned results to bounded Lipschitz domains in  $\mathbb{R}^n$ .

The paper is structured as follows. In Section 2 we define the fluid domain as well as the relevant function spaces in order to formulate a notion of weak solution for the mixed b.v.p. (1). The main result, Theorem 3.1, is stated already in Section 3, but the proof is postponed until Section 6. Section 4 is devoted to inequalities. We state three versions of Korn’s inequality, which is closely related to Nečas’ inequality. The third Korn inequality (Theorem 4.8) is a key result. So is the Strong De Rham Theorem (Corollary 5.6) in Section 5. This result follows from a theorem due to Bogovskiĭ which concerns the range of the divergence operator. We conclude this paper with a simple application of Theorem 3.1 to “pressure-driven” flow in a channel bounded by two parallel plates. We compute the solution of this problem using a finite element program and discuss its relation to the classical Poiseuille solution.

## 2. PRELIMINARIES AND NOTATION

**2.1. Euclidian structure.** Let  $\mathbb{R}^{m \times n}$  denote the set of real  $m \times n$  matrices  $X = (x_{ij})_{ij}$  equipped with the Euclidian scalar product

$$X : Y = \operatorname{tr}(X^T Y) = \sum_{j=1}^n \sum_{i=1}^m x_{ij} y_{ij}, \quad |X| = (X : X)^{1/2}$$

where  $X^T = (x_{ji})_{ij}$  in  $\mathbb{R}^{n \times m}$  denotes the transpose of  $X$ . For  $X \in \mathbb{R}^{n \times n}$  we define the symmetric part of  $X$  as

$$e(X) = \frac{1}{2} (X + X^T), \quad e_{ij}(X) = \frac{1}{2} (x_{ij} + x_{ji})$$

and the skew-symmetric part of  $X$  as

$$\omega(X) = \frac{1}{2} (X - X^T), \quad \omega_{ij}(X) = \frac{1}{2} (x_{ij} - x_{ji}).$$

The subspaces of symmetric and skew-symmetric matrices are defined as

$$\operatorname{Sym}(n) = \operatorname{Null} \omega = \{X \in \mathbb{R}^{n \times n} : \omega(X) = 0\}$$

$$\operatorname{Skew}(n) = \operatorname{Null} e = \{X \in \mathbb{R}^{n \times n} : e(X) = 0\}.$$

The identity element in  $\mathbb{R}^{n \times n}$  is denoted as  $I = (\delta_{ij})_{ij}$ . We identify  $\mathbb{R}^n$  with  $\mathbb{R}^{n \times 1}$  (column vectors) and denote the scalar product in  $\mathbb{R}^n$  as

$$x \cdot y = x^T y = \sum_{i=1}^n x_i y_i, \quad |x| = (x \cdot x)^{1/2}.$$

The standard basis vectors in  $\mathbb{R}^n$  are denoted as  $e_1, \dots, e_n$ .

**2.2. Fluid domain.** The fluid domain  $\Omega$  is assumed to be an open bounded connected subset of  $\mathbb{R}^n$  with a Lipschitz boundary  $\partial\Omega$ . By the latter condition we mean that there exists a finite collection of open sets  $U_i$  as well as corresponding rotations  $R_i \in \mathbb{R}^{n \times n}$  and  $f_i \in C^{0,1}(\mathbb{R}^{n-1})$  such that

$$\partial\Omega \subset \bigcup_i U_i \quad \text{and} \quad U_i \cap \Omega = U_i \cap R_i H_{f_i},$$

where

$$H_{f_i} = \{y \in \mathbb{R}^n : y_n > f_i(y_1, \dots, y_{n-1})\}.$$

Consequently, the outward unit normal  $\hat{n}$  is defined almost everywhere on  $\partial\Omega$  and the divergence theorem holds, i.e.

$$(3) \quad \int_{\partial\Omega} v \cdot \hat{n} \, dS = \int_{\Omega} \operatorname{div} v \, dx \quad \forall v \in C^1(\mathbb{R}^n; \mathbb{R}^n),$$

where  $dS$  denotes surface measure on  $\partial\Omega$ .  $\Gamma_D$  and  $\Gamma_N$  are assumed to be  $dS$ -measurable disjoint sets such that  $\partial\Omega = \Gamma_D \cup \Gamma_N$ . We impose further regularity assumptions on  $\Gamma_D$  and  $\Gamma_N$  below.

**2.3. Function spaces.** The dual of a Banach space  $V$  is denoted as  $V'$  and the symbol  $\langle T, v \rangle_{V', V}$ , sometimes abbreviated  $\langle T, v \rangle$  if  $V$  and  $V'$  are clear from the context, stands for the evaluation of  $T \in V'$  at  $v \in V$ . If  $Y$  is subspace of  $V$ , the quotient space  $V/Y$  is defined in the usual way and  $\{v\}$  denotes the equivalence class in  $V/Y$  to which  $v \in V$  belongs.

We shall work mainly with the following spaces

$$\begin{aligned} W &= \{v \in H^1(\Omega; \mathbb{R}^n) : v = 0 \text{ on } \Gamma_D\} \\ V &= \{v \in W : \operatorname{div} v = 0 \text{ in } \Omega\} \\ V_0 &= \{v \in H_0^1(\Omega; \mathbb{R}^n) : \operatorname{div} v = 0 \text{ in } \Omega\}. \end{aligned}$$

Clearly  $V_0 \subset V \subset W$  are closed subspaces of  $H^1(\Omega; \mathbb{R}^n)$ .

For simplicity, assume  $W$  consists of scalar valued functions. By definition,  $H^{1/2}(\Gamma_D)$  consists of all elements in  $H^{1/2}(\partial\Omega)$  restricted to  $\Gamma_D$ . We equip  $H^{1/2}(\Gamma_D)$  with the quotient norm

$$\|u\|_{H^{1/2}(\Gamma_D)} = \|\{u\}\|_{H^1(\Omega)/W} = \inf_{v \in W} \|u - v\|_{H^1(\Omega)}.$$

Moreover we define the space  $H_0^{1/2}(\Gamma_N)$  as the image of  $W$  under the trace operator:  $H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ .  $H_0^{1/2}(\Gamma_N)$  is equipped with the norm of  $W/H_0^1(\Omega)$ . The dual of  $H_0^{1/2}(\Gamma_N)$  is denoted as  $H^{-1/2}(\Gamma_N)$ . For vector valued functions we use the following useful characterization: For any  $L \in H^{-1/2}(\Gamma_N; \mathbb{R}^n)$  there exists  $G \in L^2(\Omega; \mathbb{R}^{n \times n})$  with  $\operatorname{div} G \in L^2(\Omega; \mathbb{R}^n)$  such that

$$(4) \quad \langle L, v \rangle_{H^{-1/2}(\Gamma_N), H_0^{1/2}(\Gamma_N)} = \int_{\Omega} G : \nabla v + (\operatorname{div} G) \cdot v \, dx \quad \forall v \in W.$$

By analogy with the divergence theorem we say that  $L = G\hat{n}$  on  $\Gamma_N$ , whenever (4) holds. Moreover

$$(5) \quad \|L\|_{H^{-1/2}(\Gamma_N)} = \inf \left( \int_{\Omega} |G|^2 + |\operatorname{div} G|^2 \, dx \right)^{1/2}$$

where the infimum is taken over all  $G$  satisfying (4).

We denote the space of rigid body velocities as

$$R = \{v \in H^1(\Omega; \mathbb{R}^n) : e(\nabla v) = 0 \text{ in } \Omega\}.$$

Since  $\Omega$  is connected, each element  $v$  in  $R$  can be represented as  $v(x) = Ax + b$ , for some  $A \in \operatorname{Skew}(n)$  and  $b \in \mathbb{R}^n$ .

**2.4. Weak formulation.** In view of the notation introduced above, we shall subsequently write the constitutive relation (2) as  $\sigma = -pI + 2\mu e(\nabla u)$ .

**Definition 2.1.** Assume  $f \in L^2(\Omega; \mathbb{R}^n)$ ,  $g \in L^2(\Gamma_N; \mathbb{R}^n)$  and  $h \in H^{1/2}(\Gamma_D; \mathbb{R}^n)$ . We say that  $u \in H^1(\Omega; \mathbb{R}^n)$  and  $p \in L^2(\Omega)$  are a weak

solution of (1) if  $\operatorname{div} u = 0$  in  $\Omega$ ,  $u = h$  on  $\Gamma_D$  and

$$(6) \quad \int_{\Gamma_N} g \cdot v \, dS = \int_{\Omega} (-pI + 2\mu e(\nabla u)) : \nabla v + f \cdot v \, dx$$

for all  $v$  in  $W$ .

*Derivation of weak formulation.* By formally applying the divergence theorem we obtain

$$(7) \quad \begin{aligned} \int_{\partial\Omega} \sigma \hat{n} \cdot v \, dS &= \int_{\Omega} \operatorname{div}(\sigma^T v) \, dx \\ &= \int_{\Omega} \sigma : \nabla v + (\operatorname{div} \sigma) \cdot v \, dx. \end{aligned}$$

Taking into account (1a,c) and  $v = 0$  on  $\Gamma_D$  gives (6).  $\square$

From the weak formulation (6) it is clear that the sets  $\Gamma_D$  and  $\Gamma_N$  can be defined only up to a set of measure zero. It follows that the extreme case  $\int_{\Gamma_N} dS = 0$  is equivalent to  $\Gamma_D = \partial\Omega$  and  $\Gamma_N = \emptyset$ , as  $W$  then coincides with  $H_0^1(\Omega; \mathbb{R}^n)$ . Similarly  $\int_{\Gamma_D} dS = 0$  is equivalent to  $\Gamma_D = \emptyset$  and  $\Gamma_N = \partial\Omega$ , as  $W$  then coincides with  $H^1(\Omega; \mathbb{R}^n)$ .

*Remark 2.2.* Note that it is possible to take  $g$  in the larger space  $H^{-1/2}(\Gamma_N; \mathbb{R}^n)$  provided that one replaces the surface integral of (6) with

$$\langle g, v \rangle_{H^{-1/2}(\Gamma_N), H_0^{1/2}(\Gamma_N)}.$$

### 3. MAIN RESULT

We state here the main result concerning the solvability of the b.v.p. (1). As mentioned in the introduction, the only new result is part (iii) below.

**Theorem 3.1.** *Let  $\Gamma_D$  be a closed subset of  $\partial\Omega$  and define  $\Gamma_N$  as the complement of  $\Gamma_D$  in  $\partial\Omega$ . Let  $f \in L^2(\Omega; \mathbb{R}^n)$ ,  $g \in L^2(\Gamma_N; \mathbb{R}^n)$  and  $h \in H^{1/2}(\Gamma_D; \mathbb{R}^n)$  be given functions.*

(i) *Assume  $\Gamma_D = \partial\Omega$  and*

$$(8) \quad \int_{\partial\Omega} h \cdot \hat{n} \, dS = 0.$$

*Then there exists a unique weak solution*

$$u \in H^1(\Omega; \mathbb{R}^n), \quad \{p\} \in L^2(\Omega)/\mathbb{R}$$

*of (1) such that*

$$2\mu \|u\|_{H^1(\Omega)} + \|\{p\}\|_{L^2(\Omega)/\mathbb{R}} \leq C(\|f\|_{L^2(\Omega)} + \|h\|_{H^{1/2}(\partial\Omega)}),$$

*where the constant  $C$  depends only on  $\Omega$ .*

(ii) Assume  $\Gamma_D = \emptyset$  and

$$(9) \quad \int_{\partial\Omega} g \cdot v \, dS = \int_{\Omega} f \cdot v \, dx \quad \forall v \in R.$$

Then there exists a unique weak solution

$$\{u\} \in H^1(\Omega; \mathbb{R}^n)/R, \quad p \in L^2(\Omega)$$

of (1) such that

$$2\mu \|\{u\}\|_{H^1(\Omega)/R} + \|p\|_{L^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)}),$$

where the constant  $C$  depends only on  $\Omega$ .

(iii) Assume  $\int_{\Gamma_D} dS > 0$  and  $\int_{\Gamma_N} dS > 0$ . Then there exists a unique weak solution

$$u \in H^1(\Omega; \mathbb{R}^n), \quad p \in L^2(\Omega)$$

of (1) such that

$$2\mu \|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma_N)} + \|h\|_{H^{1/2}(\Gamma_D)}),$$

where the constant  $C$  depends only on  $\Omega$  and  $\Gamma_D$ .

*Remark 3.2.* The regularity hypothesis in part (iii) of Theorem 3.1 that  $\Gamma_D$  be closed (with  $\Gamma_N$  relatively open) deserves an explanation. Clearly it is not a necessary condition. The main restriction comes from Lemma 5.1, which assumes that  $\Gamma_N$  have an interior point. Nevertheless  $\Gamma_D$  can be quite irregular, e.g. nowhere dense in  $\partial\Omega$ .

*Remark 3.3.* Note that part (i) of Theorem 3.1 holds only if  $h$  is compatible with  $\operatorname{div} u$  through (8). Similarly, part (ii) holds only if  $g$  is compatible with  $\operatorname{div} \sigma$  through (9). In contrast, part (iii) requires no such compatibility condition.

#### 4. INEQUALITIES

A fundamental result in analysis is Nečas' inequality, see [15] or [16, Lemma 7.1]. The starting point is to consider the space

$$X(\Omega) = \{T \in H^{-1}(\Omega) : \nabla T \in H^{-1}(\Omega; \mathbb{R}^n)\}.$$

Clearly  $L^2(\Omega) \subset X(\Omega)$ , but the stronger assertion  $X(\Omega) = L^2(\Omega)$  requires some regularity of  $\partial\Omega$ . In particular it is true if  $\Omega$  has a compact Lipschitz boundary, which is the present case.

**Theorem 4.1** (Nečas' inequality). *There exists a constant  $C$  depending only on  $\Omega$  such that*

$$\|p\|_{L^2(\Omega)} \leq C(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)}) \quad \forall p \in L^2(\Omega)$$

and  $X(\Omega) = L^2(\Omega)$ .

Several other important results and inequalities stated below are easily deduced from Nečas' inequality, see Tartar [20, 21] for details.

**Corollary 4.2.** *The range of  $\nabla: L^2(\Omega) \rightarrow H^{-1}(\Omega; \mathbb{R}^n)$  is equal to*

$$(\text{Null div})^\perp = \{T \in H^{-1}(\Omega; \mathbb{R}^n) : \langle T, v \rangle = 0 \quad \forall v \in V_0\}$$

Note that  $\text{Null div} = V_0$ . Since  $-\text{div}$  is the transpose of  $\nabla$  we have the following immediate corollary.

**Corollary 4.3.** *The range of  $\text{div}: H_0^1(\Omega; \mathbb{R}^n) \rightarrow L^2(\Omega)$  is equal to*

$$(\text{Null } \nabla)^\perp = \left\{ p \in L^2(\Omega) : \int_\Omega p \, dx = 0 \right\}.$$

Nečas' inequality is intimately related to Korn's inequality. We state below three versions of the Korn inequality. The third version is the most relevant one for the mixed b.v.p. (1).

**Theorem 4.4** (First Korn inequality). *There exists a constant  $K$  depending only on  $\Omega$  such that*

$$\|\nabla v\|_{L^2(\Omega)} \leq K \left( \|v\|_{L^2(\Omega)}^2 + \|e(\nabla v)\|_{L^2(\Omega)}^2 \right)^{1/2} \quad \forall v \in H^1(\Omega; \mathbb{R}^n).$$

For a proof based on Nečas' inequality, see Theorem 3.1 in Chapter III of Duvaut and Lions [5, Ch. III, Theorem 3.1]. The authors assume that  $\partial\Omega$  is of class  $C^1$ , but their proof is valid when  $\partial\Omega$  is of class  $C^{0,1}$ , as observed by Ciarlet [3].

**Theorem 4.5** (Second Korn inequality). *There exists a constant  $K$  depending only on  $\Omega$  such that*

$$\|v\|_{H^1(\Omega)/R} \leq K \|e(\nabla v)\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega; \mathbb{R}^n).$$

For a proof, see Theorem 3.4 of [5] or Theorem 2.3 (a) of [3].

To prove the third version of Korn's inequality we need an auxiliary result from classical mechanics.

**Lemma 4.6.** *Let  $\Gamma$  be a hypersurface in  $\mathbb{R}^n$  defined by*

$$\Gamma = \{x \in \mathbb{R}^n : x_n = f(x_1, \dots, x_{n-1}), \quad f \in C^{0,1}(\mathbb{R}^{n-1})\}.$$

*Let  $v \in R$  be a rigid body velocity that vanishes on a set  $E \subset \Gamma$  of positive surface measure. Then  $v$  is identically zero.*

*Proof.* From the definition of  $R$ ,  $v(x) = Ax + b$  with  $A \in \text{Skew}(n)$  and  $b \in \mathbb{R}^n$ . Since  $v$  vanishes on  $E$  we have

$$(10) \quad \int_E \phi Ax \, dS + \int_E \phi b \, dS = 0 \quad \forall \phi \in L^2(E, dS),$$

where  $dS = \sqrt{1 + |\nabla f(x')|^2} \, dx'$ ,  $x' = (x_1, \dots, x_{n-1})$ . Let  $\mathcal{P}$  be the subspace of  $L^2(E, dS)$  which is spanned by the polynomials  $p_0, \dots, p_{n-1}$  defined by

$$p_i(x) = \begin{cases} 1 & (i = 0) \\ x_i & (1 \leq i \leq n-1). \end{cases}$$



Since  $E$  has positive measure there exists a basis  $q_0, \dots, q_{n-1}$  of  $\mathcal{P}$  such that

$$(11) \quad \int_E q_i p_j dS = \delta_{ij} \quad (0 \leq i, j \leq n-1).$$

Taking  $\phi = q_0$  in (10) we deduce  $b = 0$ . Let  $y_1, \dots, y_{n-1}$  be the vectors in  $\mathbb{R}^n$  defined by

$$y_i = \int_E q_i x dS \quad (i = 1, \dots, n-1).$$

By (11),  $y_1, \dots, y_{n-1}$  are linearly independent. Taking  $\phi = q_i$  in (10) we deduce

$$Ay_i = 0 \quad (i = 1, \dots, n-1),$$

so  $\dim \text{Null } A \geq n-1$ . Since  $A$  is skew-symmetric we conclude that  $A = 0$ .  $\square$

*Remark 4.7.* If  $\Gamma_D$  has an interior point, the proof of Lemma 4.6 can be slightly simplified.

**Theorem 4.8** (Third Korn inequality). *Assume  $\int_{\Gamma_D} dS > 0$ . Then there exists a constant  $K$  depending only on  $\Omega$  and  $\Gamma_D$  such that*

$$(12) \quad \|v\|_{H^1(\Omega)} \leq K \|e(\nabla v)\|_{L^2(\Omega)} \quad \forall v \in W.$$

*Proof.* The proof is by contradiction. Suppose (12) were false. Then for each positive integer  $m$  there exists  $v_m$  in  $W$  such that  $\|v_m\|_{H^1} = 1$  and

$$(13) \quad \|e(\nabla v_m)\|_{L^2} \leq \frac{1}{m}.$$

By the Rellich-Kondrachov theorem there exists a subsequence, also denoted as  $v_m$ , that converges strongly in  $L^2(\Omega; \mathbb{R}^n)$ . Applying the first Korn inequality (Theorem 4.4) to  $v_{m+k} - v_m$ ,  $k > 0$ , we deduce from (13) that  $v_m$  is a Cauchy sequence in  $H^1(\Omega; \mathbb{R}^n)$ . Thus  $v_m$  converges strongly to some  $v$  in  $H^1(\Omega; \mathbb{R}^n)$ . Letting  $m \rightarrow \infty$  in (13) gives  $e(\nabla v) = 0$ , so we have  $v = r$  a.e. in  $\Omega$  for some  $r \in R$ . Since  $r$  vanishes on a subset of  $\partial\Omega$  of positive surface measure, Lemma 4.6 implies  $r = 0$ . This is a contradiction, because  $\|v\|_{H^1} = 1$ .  $\square$

*Remark 4.9.* Theorem 4.8 is formulated as Theorem 3.3 in [5] for  $\partial\Omega$  of class  $C^1$ . The proof presented here is merely a reproduction that proof, except for a minor detail: The implication

$$v \in R \cap \{v \in H^1(\Omega; \mathbb{R}^n) : v = 0 \text{ on } \Gamma_D\} \implies v = 0$$

provided  $\int_{\Gamma_D} dS > 0$ , is not proved in [5]. Lemma 4.6 bridges this gap.

## 5. SOME RESULTS IN VECTOR ANALYSIS

The main aim of this section is to establish a “strong” version of de Rham’s theorem, i.e. a version adapted to the b.v.p. (1). First, we establish that the space  $H_0^{1/2}(\Gamma_N; \mathbb{R}^n)$  is non-trivial under present assumptions.

**Lemma 5.1.** *Assume  $\Gamma_D$  and  $\Gamma_N$  as in Theorem 3.1(iii). Then there exist*

(i)  $\rho \in C^{0,1}(\mathbb{R}^n)$  such that

$$\rho = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \int_{\partial\Omega} \rho \, dS > 0,$$

(ii)  $v \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$  and a constant  $c > 0$  such that

$$c \leq v \cdot \hat{n} \leq 1 \quad \text{a.e. on } \partial\Omega,$$

(iii)  $\hat{v} \in H^{1/2}(\partial\Omega; \mathbb{R}^n)$  such that

$$\hat{v} = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \int_{\partial\Omega} \hat{v} \cdot \hat{n} \, dS = 1.$$

*Proof.* (i) Given  $x \in \mathbb{R}^n$ , let  $\rho(x)$  denote the distance from  $x$  to the set  $\Gamma_D$ , i.e.

$$\rho(x) = \inf_{y \in \Gamma_D} |x - y| \quad (x \in \mathbb{R}^n).$$

Clearly  $\rho \in C^{0,1}(\mathbb{R}^n)$  with Lipschitz constant equal to one. Since  $\Gamma_N$  is relatively open and non-empty, we deduce

$$\int_{\partial\Omega} \rho \, dS > 0.$$

(ii) Let  $\{U_i\}$  be the open cover used in the definition of  $\partial\Omega$  with corresponding rotations  $\{R_i\}$  and Lipschitz functions  $\{f_i\}$ . Set  $v_i = -R_i e_n$  and let  $L_i$  denote the Lipschitz constant of  $f_i$ . It is readily checked that

$$\frac{1}{\sqrt{1 + L_i^2}} \leq v_i \cdot \hat{n}(x) \leq 1$$

at each point  $x \in \partial\Omega \cap U_i$  where  $\hat{n}(x)$  is defined. Let  $\{\varphi_i\}$  be a smooth partition of unity of  $\partial\Omega$  subordinate to  $\{U_i\}$  and define

$$v(x) = \sum_i \varphi_i(x) v_i, \quad c = \min_i \frac{1}{\sqrt{1 + L_i^2}}.$$

It follows that  $c \leq v(x) \cdot \hat{n}(x) \leq 1$  for a.e.  $x \in \partial\Omega$ .

(iii) Let  $\rho$  and  $v$  be as in (i) and (ii) respectively. Clearly there exists  $\hat{v}$  of the form  $\hat{v}(x) = \rho(x)v(x)$  such that  $\int_{\partial\Omega} \hat{v} \cdot \hat{n} \, dS = 1$ .  $\square$

*Remark 5.2.* Part (ii) of Lemma 5.1 is almost as Lemma 1.5.1.9 of [7]. The proof is also the same.

*Remark 5.3.* If  $\Gamma_D = \emptyset$  there are simpler choices of  $\hat{v}$  than the above construction, e.g. one can choose  $\hat{v}$  as  $\hat{v}(x) = (n|\Omega|)^{-1}x$ . In order to choose  $\hat{v}$  as  $\rho\hat{n}$ , which is perhaps the most obvious choice,  $\partial\Omega$  must be of class  $C^{1,1}$ .

Corollary 4.3 is often attributed to Bogovskiĭ [1] (see e.g. Chapter III of [6]). We shall prove a similar statement which is more appropriate for the mixed boundary condition. Namely, that there exists a bounded linear operator  $B: L^2(\Omega) \rightarrow W/V$  such that  $\operatorname{div}(Bf) = f$ . It is therefore sometimes called a right inverse of the divergence operator.

**Theorem 5.4** (Bogovskiĭ operator). *Assume  $\Gamma_D$  and  $\Gamma_N$  as in Theorem 3.1(iii). Then for any  $f \in L^2(\Omega)$  there exists a solution  $u \in H^1(\Omega; \mathbb{R}^n)$  of the b.v.p.*

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D. \end{cases}$$

Furthermore,  $u$  is unique in  $W$  modulo  $V$  with

$$(14) \quad \|\{u\}\|_{W/V} \leq C \|f\|_{L^2(\Omega)},$$

where the constant  $C$  depends only on  $\Omega$  and  $\Gamma_D$ .

*Proof.* Given  $f \in L^2(\Omega)$ , set  $a = \int_{\Omega} f \, dx$  and choose  $\hat{v}$  as in Lemma 5.1. Since

$$\int_{\Omega} (f - a \operatorname{div} \hat{v}) \, dx = 0,$$

Corollary 4.3 asserts there exists  $w \in H_0^1(\Omega; \mathbb{R}^n)$  such that

$$f - a \operatorname{div} \hat{v} = \operatorname{div} w.$$

Thus  $u = w + a\hat{v}$  belongs to  $W$  and satisfies  $f = \operatorname{div} u$ .

Let  $A: W/V \rightarrow L^2(\Omega)$  be the linear operator defined by

$$A\{u\} = \operatorname{div} u.$$

$A$  is continuous, because

$$\|A\{u\}\|_{L^2} = \|\operatorname{div} u\|_{L^2} \leq \|u\|_{H^1} \quad \forall u \in \{u\}$$

which implies  $\|A\| \leq 1$ . Since  $A$  is one-to-one and onto, the open mapping theorem asserts that  $A^{-1}: L^2(\Omega) \rightarrow W/V$  is continuous. Choose  $u \in W$  so that  $f = \operatorname{div} u$ , which is equivalent to  $\{u\} = A^{-1}f$ . Then

$$\|\{u\}\|_{W/V} = \|A^{-1}f\|_{W/V} \leq \|A^{-1}\| \|f\|_{L^2}.$$

□

Next, we show that one can construct a divergence free “lift” of any trace in  $H^{1/2}(\Gamma_D; \mathbb{R}^n)$ .

**Corollary 5.5** (Lift operator). *Assume  $\Gamma_D$  and  $\Gamma_N$  as in Theorem 3.1(iii). For any  $h \in H^{1/2}(\Gamma_D; \mathbb{R}^n)$  there exists  $u \in H^1(\Omega; \mathbb{R}^n)$  such that*

$$(15) \quad \begin{cases} \operatorname{div} u = 0 & \text{in } \Omega \\ u = h & \text{on } \Gamma_D. \end{cases}$$

Moreover  $u$  can be chosen so that

$$(16) \quad \|u\|_{H^1(\Omega)} \leq C \|h\|_{H^{1/2}(\Gamma_D)}$$

where the constant  $C$  depends only on  $\Omega$  and  $\Gamma_D$ .

*Proof.* Choose any  $v \in H^1(\Omega; \mathbb{R}^n)$  such that  $v = h$  on  $\Gamma_D$ . Since  $\operatorname{div} v \in L^2(\Omega)$ , Theorem 5.4 says there exists  $w \in H^1(\Omega; \mathbb{R}^n)$  such that  $\operatorname{div} w = \operatorname{div} v$  and  $w = 0$  on  $\Gamma_D$ . Set  $u = v - w$ . Then  $\operatorname{div} u = 0$  in  $\Omega$  and  $u = h$  on  $\Gamma_D$ . Moreover

$$\|u\|_{H^1} \leq \|v\|_{H^1} + \|w\|_{H^1} \leq \|v\|_{H^1} + C \|\operatorname{div} v\|_{L^2} \leq (C + 1) \|v\|_{H^1}.$$

Taking the infimum over all  $v$  such that  $v = h$  on  $\partial\Omega$  gives (16).  $\square$

Corollary 4.2 is usually referred to as “De Rham’s Theorem” [11, 22]. In alternative language, it says that any  $L$  in  $V_0^\perp$  can be represented as

$$\langle L, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_{\Omega} p \operatorname{div} v \, dx$$

for some  $p$  in  $L^2(\Omega)$ . Observe that  $p$  is only unique up to an additive constant.

The following result is called the “Strong De Rham Theorem” because the conclusion is stronger, compared with the classical De Rham theorem. This is not surprising as the hypothesis is stronger.

**Corollary 5.6** (Strong De Rham). *Assume  $\Gamma_D$  and  $\Gamma_N$  as in Theorem 3.1(iii). Suppose  $L$  in  $W'$  satisfies*

$$\langle L, v \rangle_{W', W} = 0 \quad \forall v \in V.$$

Then there exists a unique  $p$  in  $L^2(\Omega)$  such that

$$\langle L, v \rangle_{W', W} = \int_{\Omega} p \operatorname{div} v \, dx \quad \forall v \in W.$$

Moreover

$$\|p\|_{L^2(\Omega)} \leq C \|L\|_{W'}$$

where  $C$  is the constant of Theorem 5.4.

*Proof.* Since  $L$  belongs to  $V^\perp$  it can be identified with an element in  $(W/V)'$  (also denoted as  $L$ ) through

$$\langle L, \{v\} \rangle_{(W/V)', (W/V)} = \langle L, v \rangle_{W', W}.$$

Let  $B: L^2(\Omega) \rightarrow W/V$  be the Bogovskii operator. Then

$$\langle L, Bf \rangle = \langle B^T L, f \rangle \quad \forall f \in L^2(\Omega),$$

where  $B^T: (W/V)' \rightarrow L^2(\Omega)$  is the transpose of  $B$ . Since  $B$  is an isomorphism it follows that

$$\langle L, v \rangle = \int_{\Omega} (B^T L) \operatorname{div} v \, dx \quad \forall v \in W.$$

The result now follows with  $p = B^T L$ .  $\square$

*Remark 5.7.* We shall apply Corollary 5.6 when  $L$  is defined by

$$\langle L, v \rangle = \int_{\Omega} F : \nabla v + f \cdot v \, dx,$$

where  $F \in L^2(\Omega; \mathbb{R}^{n \times n})$  and  $f \in L^2(\Omega; \mathbb{R}^n)$  are given functions. If  $L$  vanishes on  $V$ , Corollary 5.6 says there exists a unique  $p \in L^2(\Omega)$  such that

$$\begin{cases} \operatorname{div}(-p I + F) = f & \text{in } \Omega \\ (-p I + F)\hat{n} = 0 & \text{on } \Gamma_N. \end{cases}$$

## 6. PROOF OF MAIN RESULT

We give here the proof of Theorem 3.1(iii), i.e. for the case that

$$\int_{\Gamma_D} dS > 0 \quad \text{and} \quad \int_{\Gamma_N} dS > 0.$$

For the proof of Theorem 3.1(ii) we refer to Chapter IV of [2]. Note that in view of Remark 5.3  $\partial\Omega$  need not be of class  $C^{1,1}$ , as stated in [2]. For simplicity we take  $\mu = 1/2$ .

Using the results proved in Sections 4–5, notably Theorem 4.8 and Corollary 5.6, the existence and uniqueness of weak solutions of (1) follows from standard applications of the Lax–Milgram theorem. It is only for the reader's convenience that we provide all details. To this end, consider the symmetric bilinear form

$$a(u, v) = \int_{\Omega} e(\nabla u) : e(\nabla v) \, dx,$$

which satisfies

$$\begin{aligned} |a(u, v)| &\leq \|u\|_{H^1} \|v\|_{H^1} \\ a(u, u) &\geq K^{-1} \|u\|_{H^1}^2 \end{aligned} \quad (u, v \in W),$$

where  $K > 0$  is the constant in the Korn inequality (12). According to the Lax–Milgram theorem, for any  $L \in V'$  there exists a unique element  $u$  in  $V$  such that

$$(17) \quad a(u, v) = \langle L, v \rangle_{V', V} \quad \forall v \in V.$$

Standard estimations give

$$\begin{aligned} \|e(\nabla u)\|_{L^2} &\leq K \|L\|_{V'} \\ \|u\|_{H^1} &\leq K^2 \|L\|_{V'}. \end{aligned}$$

For subsequent estimations of the pressure we introduce the space

$$J = \{w \in W : a(w, v) = 0 \quad \forall v \in V\}.$$

In other words,  $J$  is the orthogonal complement of  $V$  in  $W$  with respect to the scalar product  $a(\cdot, \cdot)$ .

*Proof of Theorem 3.1(iii).* Using the linearity of the system (1) we shall construct  $u$  and  $p$  as superpositions in four steps.

1. Assume (temporarily) that  $g = 0$  and  $h = 0$ . Then  $u \in V$  and (6) implies

$$(18) \quad \int_{\Omega} e(\nabla u) : \nabla v + f \cdot v \, dx = 0 \quad \forall v \in V.$$

Clearly (18) follows from (17) by taking

$$(19) \quad \langle L, v \rangle = - \int_{\Omega} f \cdot v \, dx$$

which satisfies  $\|L\|_{V'} \leq \|f\|_{L^2}$ . The best estimate, however, is given by

$$(20) \quad \|e(\nabla u)\|_{L^2} \leq \frac{1}{\sqrt{\lambda_1}} \|f\|_{L^2}, \quad \lambda_1 = \inf_{\substack{v \in V \\ \|v\|_{L^2}=1}} a(v, v).$$

Applying the Strong De Rham Theorem (Corollary 5.6) to (18), we obtain a unique  $p \in L^2(\Omega)$  such that

$$(21) \quad \int_{\Omega} (-p I + e(\nabla u)) : \nabla v + f \cdot v \, dx = 0 \quad \forall v \in W.$$

The corresponding weak formulation for the pressure is

$$(22) \quad \int_{\Omega} -p \operatorname{div} v + f \cdot v \, dx = 0 \quad \forall v \in J.$$

Observe that by taking  $v \in J$  we have eliminated the viscous term  $e(\nabla u)$ . To estimate  $p$ , choose  $v \in J$  so that  $\operatorname{div} v = p$  as in Theorem 5.4. We deduce

$$(23) \quad \|p\|_{L^2} \leq C \|f\|_{L^2}.$$

2. Assume  $f = 0$  and  $h = 0$ . Then  $u \in V$  and (6) implies

$$(24) \quad \int_{\Omega} e(\nabla u) : \nabla v = \int_{\Gamma_N} g \cdot v \, dS \quad \forall v \in V.$$

The existence of a  $u$  satisfying (24) follows from (17) by choosing

$$(25) \quad \langle L, v \rangle = \int_{\Gamma_N} g \cdot v \, dS.$$

Clearly  $L$  defines an element in  $H^{-1/2}(\Gamma_N; \mathbb{R}^n)$ . Using the representation (4) we obtain

$$(26) \quad \int_{\Gamma_N} g \cdot v \, dS = \int_{\Omega} G : \nabla v + (\operatorname{div} G) \cdot v \, dx \quad \forall v \in W,$$

for some  $G \in L^2(\Omega; \mathbb{R}^{n \times n})$  with  $\operatorname{div} G \in L^2(\Omega; \mathbb{R}^n)$ . In other words  $g = G\hat{n}$  on  $\Gamma_N$ . Clearly  $\|L\|_{V'} \leq \|g\|_{H^{-1/2}(\Gamma_N)}$ . Writing (24) as

$$(27) \quad \int_{\Omega} (e(\nabla u) - G) : \nabla v - \operatorname{div} G \cdot v \, dx = 0 \quad \forall v \in V$$

and invoking the strong de Rham theorem, gives a unique  $p$  in  $L^2(\Omega)$  such that

$$(28) \quad \int_{\Omega} (-pI + e(\nabla u) - G) : \nabla v - \operatorname{div} G \cdot v \, dx = 0 \quad \forall v \in W.$$

Taking  $v \in J$  such that  $\operatorname{div} v = p$ , eliminates  $e(\nabla u)$  and implies the estimate

$$(29) \quad \|p\|_{L^2} \leq C (\|G\|_{L^2}^2 + \|\operatorname{div} G\|_{L^2}^2)^{1/2}.$$

Taking the infimum over all  $G$  satisfying (26) gives  $\|p\|_{L^2} \leq C \|g\|_{H^{-1/2}}$ .

3. Assume  $f = 0$  and  $g = 0$ . Then (6) implies

$$(30) \quad \int_{\Omega} e(\nabla u) : \nabla v \, dx = 0 \quad \forall v \in V.$$

Since  $u$  is now assumed to satisfy  $u = h$  on  $\Gamma_D$  we seek  $u$  in the admissible class

$$V_h = \{v \in H^1(\Omega; \mathbb{R}^n) : \operatorname{div} v = 0 \text{ in } \Omega, \quad v = h \text{ on } \Gamma_D\}.$$

By Corollary 5.5,  $V_h$  is not empty. Since  $V_h$  is a closed convex set, there exists a unique  $u \in V_h$  such that

$$(31) \quad \int_{\Omega} |e(\nabla u)|^2 \, dx \leq \int_{\Omega} |e(\nabla v)|^2 \, dx \quad \forall v \in V_h.$$

It is readily checked that (31) is equivalent to (30). By Korn's inequality and (31) we have

$$\|u - v\|_{H^1} \leq K \|e(\nabla u) - e(\nabla v)\|_{L^2} \leq 2K \|e(\nabla v)\|_{L^2}.$$

for any  $v \in V_h$ . Thus

$$\|u\|_{H^1} \leq \|u - v\|_{H^1} + \|v\|_{H^1} \leq (2K + 1) \|v\|_{H^1}.$$

Choosing  $v$  as in Corollary 5.5 we deduce

$$\|u\|_{H^1} \leq C \|h\|_{H^{1/2}},$$

where the constant  $C$  depends only on  $\Omega$  and  $\Gamma_D$ . Applying the strong de Rham theorem to (30) gives a unique  $p \in L^2(\Omega)$  such that

$$(32) \quad \int_{\Omega} (-pI + e(\nabla u)) : \nabla v \, dx = 0 \quad \forall v \in W$$

with

$$(33) \quad \|p\|_{L^2} \leq C \|e(\nabla u)\|_{L^2} \leq C \|h\|_{H^{1/2}}.$$

4. Let  $(u_f, p_f)$ ,  $(u_g, p_g)$  and  $(u_h, p_h)$  denote the solutions of (21), (28) and (32) respectively. Then

$$u = u_f + u_g + u_h, \quad p = p_f + p_g + p_h$$

solve the weak formulation (6). Moreover, the above estimates give

$$\|u\|_{H^1} + \|p\|_{L^2} \leq C(\|f\|_{L^2} + \|g\|_{H^{-1/2}} + \|h\|_{H^{1/2}}).$$

This estimate implies the uniqueness of  $u$  and  $p$ .  $\square$

7.  $\square$

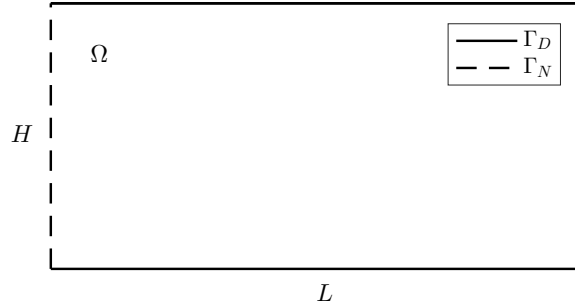


FIGURE 1. Boundary of rectangular fluid domain  $\Omega$

We choose a simple application to illustrate Theorem 3.1(iii), namely the two-dimensional steady flow between two parallel plates driven by a “pressure gradient”. There seems to be no general consensus as to the meaning of the term “pressure-driven flow” in the realm of fluid dynamics. By pressure-driven flow, we mean the flow induced by a *normal stress distribution* alone. For alternative meanings we refer to [4] and [8].

Let  $\Omega \subset \mathbb{R}^2$  be the rectangular domain  $(0, L) \times (0, H)$ , i.e.

$$\Omega = \{(x_1, x_2) : 0 < x_1 < L, \quad 0 < x_2 < H\}$$

with

$$(34) \quad \Gamma_N = \Gamma_1 \cup \Gamma_2, \quad \Gamma_D = \Gamma_3 \cup \Gamma_4,$$

where

$$\begin{aligned} \Gamma_1 &= \{(x_1, x_2) : x_1 = 0, \quad 0 < x_2 < H\} \\ \Gamma_2 &= \{(x_1, x_2) : x_1 = L, \quad 0 < x_2 < H\} \\ \Gamma_3 &= \{(x_1, x_2) : x_2 = 0, \quad 0 \leq x_1 \leq L\} \\ \Gamma_4 &= \{(x_1, x_2) : x_2 = H, \quad 0 \leq x_1 \leq L\}. \end{aligned}$$

See Figure 1 for an illustration.



Assume  $f = 0$ ,  $h = 0$  in (1) and define

$$(35) \quad g(x) = \begin{cases} -p_{\text{in}} \hat{n} & \text{if } x \in \Gamma_1 \\ -p_{\text{out}} \hat{n} & \text{if } x \in \Gamma_2, \end{cases}$$

where  $p_{\text{in}}$  (inlet pressure) and  $p_{\text{out}}$  (outlet pressure) are given constants. Thus, we have a normal stress condition on the lateral boundaries. According to Theorem 3.1(iii) the b.v.p. (1) has a unique solution  $u \in H^1(\Omega; \mathbb{R}^2)$ ,  $p \in L^2(\Omega)$  such that

$$\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C \|g\|_{H^{-1/2}(\Gamma_N)}.$$

We compute  $u$  and  $p$  using the “Creeping Flow” module in Comsol Multiphysics, which is a software based on the finite element method, for

$$L = 2, \quad H = 1, \quad p_{\text{in}} = 1, \quad p_{\text{out}} = 0, \quad \mu = 1.$$

It is instructive to compare  $(u, p)$  with the well-known Poiseuille solution  $(\tilde{u}, \tilde{p})$  (see e.g. Chapter 7 of [18]) defined by

$$(36) \quad \begin{cases} \tilde{u}(x) = -\frac{x_2(H - x_2)}{2\mu} \nabla \tilde{p}(x) \\ \tilde{p}(x) = p_{\text{out}} \frac{x_1}{L} + p_{\text{in}} \left(1 - \frac{x_1}{L}\right) \end{cases} \quad (x \in \Omega).$$

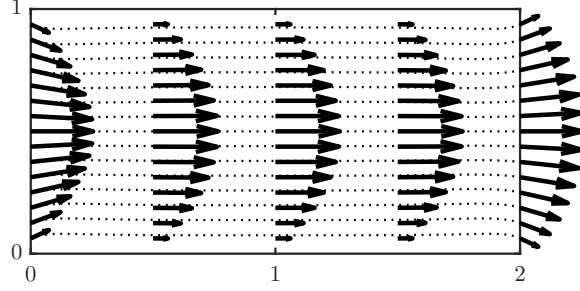
Observe that  $(\tilde{u}, \tilde{p})$  also satisfies (1a,b,d) but not (1c) as

$$\tilde{p} = \begin{cases} p_{\text{in}} & \text{on } \Gamma_1 \\ p_{\text{out}} & \text{on } \Gamma_2, \end{cases}$$

$\tilde{u}_2 = 0$  and

$$\begin{aligned} 2\mu e(\nabla \tilde{u}) &= \mu \begin{bmatrix} 2\frac{\partial \tilde{u}_1}{\partial x_1} & \frac{\partial \tilde{u}_1}{\partial x_2} + \frac{\partial \tilde{u}_2}{\partial x_1} \\ \frac{\partial \tilde{u}_2}{\partial x_1} + \frac{\partial \tilde{u}_1}{\partial x_2} & 2\frac{\partial \tilde{u}_2}{\partial x_2} \end{bmatrix} \\ &= \frac{(H - 2x_2)(p_{\text{in}} - p_{\text{out}})}{2L} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Thus  $(u, p)$  and  $(\tilde{u}, \tilde{p})$  are not identical. Nevertheless the velocity profiles of  $u$  and  $\tilde{u}$  are similar (compare Figures 2 and 3) as are the boundary stresses (compare Figures 4 and 5). Note that, for the sake of visibility, velocity vectors are scaled by a factor 4.0 while stress vectors are scaled by a factor 0.25. The pressure distribution  $p$  behaves like the linear function  $\tilde{p} = \tilde{p}(x_1)$  around  $x_1 = 1$  but deviates more and more from  $\tilde{p}$  as one approaches the lateral boundaries (see Figures 6 and 7). Note also that the viscous stresses on  $\Gamma_N$  are  $x_1$ -directional in Figure 8 while they are  $x_2$ -directional in Figure 9.

FIGURE 2. Velocity  $u$  (dotted streamlines)

To summarize, our numerical example suggests that the Poiseuille solution  $(\tilde{u}, \tilde{p})$  defined by (36) *approximates* the normal stress solution  $(u, p)$ . In fact, since

$$2\mu e(\nabla \tilde{u})\hat{n} = O\left(\frac{H}{L}\right)$$

one may expect them to be asymptotically equivalent as  $H/L \rightarrow 0$ , i.e. if  $\Omega$  is “infinitely long” or “infinitely thin”. A precise statement, when  $H \rightarrow 0$  and  $L$  is constant, follows.

**Theorem 7.1.** *Let  $\Omega$  be the rectangle  $(0, L) \times (0, H)$  with  $\Gamma_N$  and  $\Gamma_D$  as in (34). Let  $(u, p)$  be the solution of (1) with  $f = 0$ ,  $h = 0$  and*

$$g(x) = -\tilde{p}(x)\hat{n}, \quad \tilde{p}(x) = p_{\text{out}} \frac{x_1}{L} + p_{\text{in}} \left(1 - \frac{x_1}{L}\right) \quad (x \in \Gamma_N).$$

*Then*

$$\begin{aligned} \lim_{H \rightarrow 0} \frac{1}{|\Omega|} \int_{\Omega} \frac{L}{H^2} u(x) \phi\left(\frac{x_1}{L}, \frac{x_2}{H}\right) dx &= \int_{\square} u^0(y) \phi(y) dy \\ \lim_{H \rightarrow 0} \frac{1}{|\Omega|} \int_{\Omega} p(x) \phi\left(\frac{x_1}{L}, \frac{x_2}{H}\right) dx &= \int_{\square} p^0(y) \phi(y) dy \end{aligned}$$

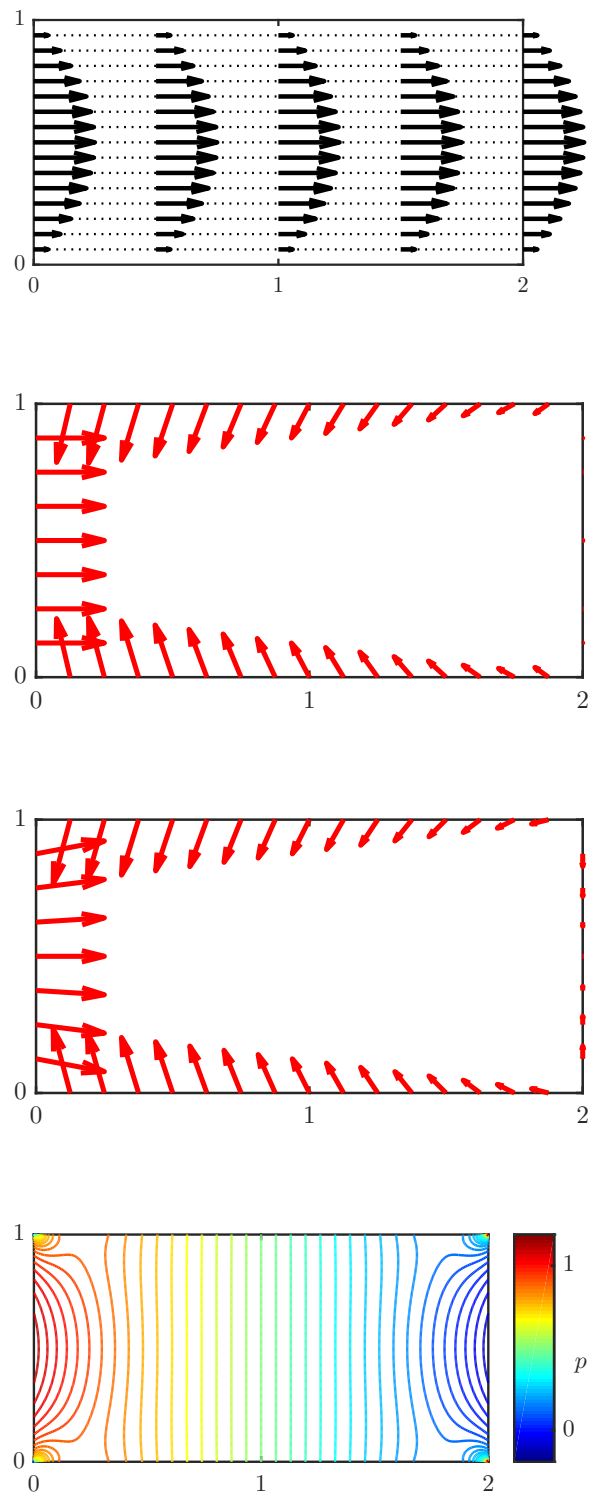
*for all  $\phi \in C(\mathbb{R}^2)$ , where  $\square$  is the unit square  $(0, 1) \times (0, 1)$  and*

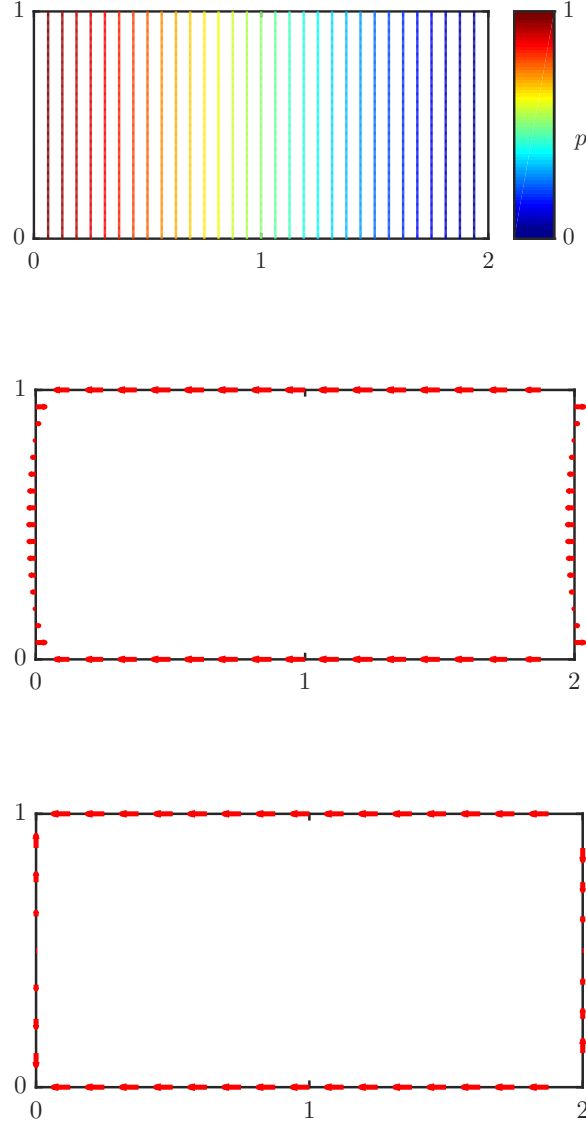
$$\begin{cases} u^0(y) = -\frac{y_2(1-y_2)}{2\mu} \nabla p^0(y) \\ p^0(y) = p_{\text{out}} y_1 + p_{\text{in}} (1 - y_1) \end{cases} \quad (y \in \square).$$

We shall not prove Theorem 7.1 here as a more general result, connected with lubrication theory, will appear in a forthcoming paper. The notion of convergence used in Theorem 7.1 is called “two-scale convergence for thin domains”. It was introduced by Marušić and Marušić-Paloka in [12].

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FIGURE 6. Pressure  $p$

FIGURE 9. Viscous stress  $2\mu e(\nabla\tilde{u})\hat{n}$ 

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